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The Morse–Smale Property for a Semi-linear Parabolic Equation

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1. INTRODUCTION

In this paper we shall study the dynamics of semilinear parabolic PDE's of the following type:

$$\begin{aligned} &u_t = u_{xx} + f(x, u) && 0 < x < 1,\ t > 0, \\ &u(0, t) = u(1, t) = 0 && t > 0, \\ &u(x, 0) = \phi(x). \end{aligned} \tag{I}$$

We make the following assumptions about the smoothness of the non-linear source term $f(x, u)$:

(f1) f, f_u, f_{uu}, and f_{ux} are continuous on $[0, 1] \times \mathbb{R}$.

About the way $f(x, u)$ may grow as $|u| \to \infty$ we assume:

(f2) $\exists K > 0\colon |f(x, u)| \leqslant K(1 + |u|)$ for all $(x, u) \in [0, 1] \times \mathbb{R}$.

It is well known that Problem I has a unique classical solution $u(x, t)$ for any initial value $\phi(x)$ belonging to the Hilbert space

$$X = H_0^1(0, 1).$$

See Henry's book [6] for details. In fact the mappings $T(t)\colon X \to X$ defined by

$$(T(t)\,\phi)(x) = u(t, x),$$

where $u(t, x)$ is the classical solution of Problem I with initial value $\phi(x)$, form a semiflow on X. Each map $T(t)$ has a Fréchet derivative $DT(t)$, and

$$(t, \phi, \psi) \mapsto (DT(t)\,\phi) \cdot \psi$$

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is continuous from $[0, \infty) \times X \times X$ to X. Here $DT(t)\,\phi$ is the derivative of $T(t)$ at ϕ. Again we refer to [6] for more details.

Our main result is that, if ϕ and ψ are hyperbolic fixed points of the semiflow $T(t)$, then the stable manifold of ϕ intersects the unstable one of ψ transversally.

This implies that any semiflow $T(t)$ that has a finite number of fixed points, all of which are hyperbolic, is a Morse–Smale system in the sense of Hale, Magelhães, and Oliva [5]. It is shown in [5] that such systems are structurally stable.

The proof of our result relies on a theorem of Matano. This theorem says that if $u(x, t)$ is a solution of a linear parabolic equation, then the number of sign changes in the x direction of $u(x, t)$ cannot increase with time (see Sect. 3). Matano's results have been used by other authors (Matano [9], Hale [4]) to study semilinear parabolic equations. The difference between our approach and the foregoing ones is that we apply Matano's ideas to the derivative $DT(t)$ of $T(t)$ instead of to $T(t)$ it self.

The organization of this paper is the following. In Section two we collect some basic facts about the semiflow $T(t)$, its derivative $DT(t)$, and the stable and unstable manifolds of hyperbolic points of $T(t)$. In Section 3 we use Matano's theorem to derive precise information about solutions which lie entirely on the stable or unstable manifold of a fixed point. The main result is proved in Section 4.

2. The Derivative $DT(t)$

For given $\phi \in X$ the solution of Problem I with initial value ϕ is given by $u(t) = T(t)\,\phi$. If we change the initial value ϕ slightly, by ψ, then the orbit $u(t)$ will generally change slightly, say by $v(t)$. This change is measured by the derivative $DT(t)\,\phi$.

For any $\psi \in X$ there is a curve $v(t)$ in X given by

$$v(t) = (DT(t)\,\phi) \cdot \psi.$$

This curve represents the classical solution $v(t, x)$ of the linearized equation

$$\begin{aligned} &v_t = v_{xx} + f_u(x, u(x, t))\,v && 0 < x < 1,\ t > 0, \\ &v(0, t) = v(1, t) = 0 && t > 0, \\ &v(x, 0) = \psi(x) && 0 < x < 1. \end{aligned} \tag{II}$$

Note that unless ϕ is a fixed point of the semigroup $T(t)$, Problem II is not autonomous. Actually one should speak of the curve $(u(t), v(t))$ in the tangent bundle TX of X. However we shall identify TX with $X \times X$.

LEMMA 1. *For any $\phi \in X$ and $t > 0$ the derivative $DT(t)\,\phi$ satisfies*

(a) $\operatorname{Ker}(DT(t)\,\phi) = (0)$

(b) $\operatorname{Range}(DT(t)\,\phi)$ *is dense in X.*

Proof. The first statement is equivalent to backward uniqueness for Problem II which is known to hold (see [1, 8, 9]). In [6] Henry shows that (b) follows from backward uniqueness for the adjoint equation (apply his Theorem 7.3.3, with $\alpha = 0$, $\beta = \frac{1}{2}$, and $X = L_2(0, 1)$).

Consider a fixed point $\phi \in X$ of the semiflow. From the chain rule it then follows that

$$U(t) = DT(t)\,\phi$$

is a strongly continuous one parameter semi-group on X. Its generator is easily seen to be

$$A = \left(\frac{d}{dx}\right)^2 + f_u(x, \phi(x))$$

$$D(A) = H^2(0, 1) \cap H_0^1(0, 1).$$

Thus its spectrum is given by

$$\sigma(U(t)) = \{0 < \cdots < e^{\lambda_2 t} < e^{\lambda_1 t} < e^{\lambda_0 t}\},$$

where $\{\lambda_0 > \lambda_1 > \lambda_2 > \cdots\}$ are the eigenvalues of A.

The fixed point is said to be hyperbolic if for all $t > 0$, $\sigma(U(t))$ is disjoint from the unit circle. Hence ϕ is hyperbolic if zero is not an eigenvalue of A.

In this case only a finite number of eigenvalues of A is positive, say k. This number $k \geqslant 0$ is called the *Morse-index* of ϕ, or just the index for short.

Let $v_j \in X$ be the eigenvector of A with eigenvalue λ_j $(j = 0, 1, 2,...)$. We denote the subspace of X spanned by the first k eigenvectors by X_+, and the closed subspace generated by the other eigenfunctions by X_-. We then have the following $U(t)$ invariant splitting of X: $X = X_+ \oplus X_-$.

The global stable and unstable manifolds are defined as follows:

$$W^s(\phi) = \{\eta \in X\colon T(t)\,\eta \to \phi \text{ as } t \to \infty\}$$

$$W^u(\phi) = \{\eta \in X\colon \text{there exists an orbit } u\colon (-\infty, 0] \to X \text{ such that}$$
$$u(0) = \eta \text{ and } u(t) \to \phi \text{ as } t \to -\infty\}.$$

PROPOSITION 1. *$W^s(\phi)$ and $W^u(\phi)$ are imbedded submanifolds of X. Furthermore*

$$T_\phi W^s(\phi) = X_-$$
$$T_\phi W^u(\phi) = X_+$$

holds, and in particular $\dim W^u(\phi) = k$, *the index of* ϕ.

This is proved in [6, Theorems 6.1.9 and 6.1.10]. Lemma 1 and the gradient structure of the semiflow play an important role in the proof.

3. MATANO'S PRINCIPLE

For a given $\phi \in C([0, 1])$ we define the number of sign changes

$$\mathscr{S}(\phi) = \sup(k \geqslant 0\text{: There exist } 0 < t_0 < t_1 < \cdots < t_k < 1 \text{ such that } \phi(t_j)\, \phi(t_{j-1}) < 0\ j = 1, 2, \ldots, k).$$

When the supremum does not exist, we shall define $\mathscr{S}(\phi)$ to be infinite.

Matano's principle can now be formulated as follows:

PROPOSITION 2. *Let $u(x, t)$ be a classical solution of*

$$u_t = a(x, t)\, u_{xx} + b(x, t)\, u_x + c(x, t)\, u, \qquad 0 < x < 1,\ 0 < t < T,$$
$$u(0, t) = u(1, t) = 0, \qquad 0 < t < T.$$

Here a, b, and c are continuous on $[0, 1] \times [0, T]$ and there exists a positive constant M such that

$$M^{-1} \leqslant a(x, t) \leqslant M, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T,$$

i.e., the equation is uniformly parabolic. If one defines $\phi(x) = u(x, 0)$ and $\psi(x) = u(x, T)$ then

$$\mathscr{S}(\psi) \leqslant \mathscr{S}(\phi).$$

In other words, the number of sign changes of a solution of a linear parabolic equation cannot increase with time. A proof of this fact can be found in [9, Lemma 2.6].

This principle can be applied to the linear equation which defines $DT(t)\,\phi$. We then get

LEMMA 2. *Let $\phi, \psi_0 \in X$ be given. Define*

$$\psi(t) = (DT(t)\,\phi) \cdot \psi_0.$$

Then $\mathscr{S}(\psi(t))$ is nonincreasing in t.

Proof. The curve $\psi(t)$ represents a classical solution $v(t, x)$ of Problem II, to which Matano's principle is directly applicable.

We shall now explore some of the consequences of this lemma for the stable and unstable manifolds of a fixed point of $T(t)$.

Let $\phi \in X$ be a hyperbolic fixed point of $T(t)$, and let W^u and W^s be its unstable and stable manifolds. Recall that the Morse index of ϕ, defined in Section 2, is the dimension of W^u.

LEMMA 3. *Let N, the Morse index of ϕ, be positive. For each $\psi \in W^u$, and $k = 1, \ldots, N$ there exists a k-dimensional linear subspace $L_k(\psi)$ of the tangent space of W^u at ψ, $T_\psi W^u$, such that*

$$\forall \chi \in L_k(\psi) \setminus \{0\} : \mathscr{S}(\chi) < k.$$

In particular, if $\chi \in T_\psi W^u$ then $\mathscr{S}(\chi) < N$.

Proof. The unstable manifold W^u is diffeomorphic to $\mathbb{R}^N$, and the semiflow W^u is conjugate to that generated by a C^1 vectorfield on $\mathbb{R}^N$, say $F: \mathbb{R}^N \to \mathbb{R}^N$.

If we identify ϕ with the origin in $\mathbb{R}^N$, then the linear part of F at ϕ, $F'(0)$, has eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_{N-1} > 0$. Let A and v_j be defined as in Section 2. Note that, since $T(t)$ is given by an ODE on W^u, the semiflow $T(t)$ can be defined for $t < 0$. Now define

$$L_k(\psi) = \{0\} \cup \left\{ \eta \in T_\psi W^u \mid \eta \neq 0, \text{ and } \eta \text{ satisfies} \right.$$
$$\lim_{t \to -\infty} \frac{(DT(t)\,\psi)\,\phi}{\|(DT(t) \cdot \psi) \cdot \phi\|} = v_j \tag{3.1}$$
$$\left. \text{for some } j < k \right\}.$$

It can be shown that W^u is a C^1 submanifold of $H^s \cap H_0^1$ for $1 \leqslant s < 2$, so we may assume that the limit in (3.1) is taken in $C^1([0, 1])$.

It is a well-known fact from the asymptotic theory of linear ODE's that $L_k(\psi)$ is a k-dimensional subspace of $T_\psi W^u$. Furthermore, for any nonzero η in $L_k(\psi)$ there is a $j < k$ such that (3.1) holds. Hence, for large enough t $(DT(-t)\,\psi) \cdot \eta$ has j sign changes. Applying Lemma 2 we see that $\mathscr{S}(\eta) \leqslant j < k$ holds for all $\eta \in L_k(\psi)$.

Next we shall prove a similar statement about the stable manifold. Because this manifold is not finite dimensional, we have to be more careful about the asymptotics of the linearized equation. The hard parts of the proof will be postponed to the Appendix.

LEMMA 4. *Let $\psi \in W^s$ and $\chi \in T_\psi W^s \setminus \{0\}$ be given. Then $\mathcal{S}(\psi) \geqslant N$.*

Proof. Define $u(t) = T(t)\psi$ and $v(t) = (DT(t)\psi)\cdot\chi$. Let A and v_j be defined as in Section 2. It is shown in the Appendix (see Lemma 7 and the subsequent discussion) that for some $j \geqslant 0$ we have

$$\frac{v(t)}{\|v(t)\|} \to v_j \qquad \text{as} \quad t \to \infty.$$

For each $t \geqslant 0$ the point

$$w(t) = \left(u(t), \frac{v(t)}{\|v(t)\|}\right) \in TX = X \times X$$

lies in the tangent bundle TW^s of W^s. This is a locally closed subset of $X \times X$, so the limit $w(\infty) = (\phi, v_j)$ also lies in this bundle. Hence $v_j \in T_\phi W^s$, so $j \geqslant N$ (remember that $T_\phi W^s$ is spanned by $\{v_N, v_{N+1}, v_{N+2}, ...\}$!).

Since $v(t)/\|v(t)\|$ converges in X and therefore in $C([0, 1])$ to v_j we have for large enough $t > 0$:

$$\mathcal{S}(v(t)) \geqslant \mathcal{S}(v_j) = j \geqslant N.$$

Matano's principle then implies that

$$\mathcal{S}(\chi) = \mathcal{S}(v(0)) \geqslant \mathcal{S}(v(t)) \geqslant N.$$

4. TRANSVERSALITY

In this section we state and prove our main result.

THEOREM. *Let ϕ and ψ be hyperbolic fixed points of the semigroup $T(t)$ determined by Problem I. Then the stable manifold of ϕ, $W^s(\phi)$ and the unstable manifold of ψ, $W^u(\psi)$, intersect transversally:*

$$W^s(\phi) \pitchfork W^u(\psi).$$

In particular, if the intersection is nonempty then $\operatorname{index}(\psi) > \operatorname{index}(\phi)$.

Proof. If $W^s(\phi)$ and $W^u(\psi)$ do not intersect, then $W^s(\phi) \pitchfork W^u(\psi)$ trivially. So we assume that there is a point of intersection,

$$u_0 \in W^s(\phi) \cap W^u(\psi).$$

Then there exists a connecting orbit $u\colon \mathbb{R} \to X$ such that

$$u(-\infty) = \psi, \qquad u(+\infty) = \phi, \qquad \text{and} \qquad u(0) = u_0.$$

To prove that $W^s(\phi) \pitchfork W^u(\psi)$ at u_0 we have to show that

$$T_{u_0} W^s(\phi) + T_{u_0} W^u(\psi) = X$$

holds. First note that the orbit $u(t)$ is a C^1 curve which lies in the intersection of $W^s(\phi)$ and $W^u(\psi)$. This implies that

$$u_t(0) \in T_{u_0} W^s(\phi) \cap T_{u_0} W^u(\psi).$$

Since $u_t(0)$ is nonzero Lemmas 3 and 4 imply that

$$\text{index}\, \phi < \mathscr{S}(u_t(0)) \leqslant \text{index}\, \psi.$$

Set $N = \text{index}\, \phi$. From Lemma 3 it follows that there exists an N-dimensional subspace

$$L_N(u_0) \subset T_{u_0} W^u(\psi)$$

such that for any nonzero $\chi \in L_N(u_0)$ one has

$$\mathscr{S}(\chi) < N.$$

On the other hand, Lemma 2 tells us that for every nonzero $\chi \in T_{u_0} W^s(\phi)$,

$$\mathscr{S}(\chi) \geqslant N$$

holds. This shows that

$$L_N(u_0) \cap T_{u_0} W^s(\phi) = \{0\},$$

and because

$$\dim L_N(u_0) = N = \operatorname{codim} T_{u_0} W^s(\phi),$$

we have

$$L_N(u_0) \oplus T_{u_0} W^s(\phi) = X$$

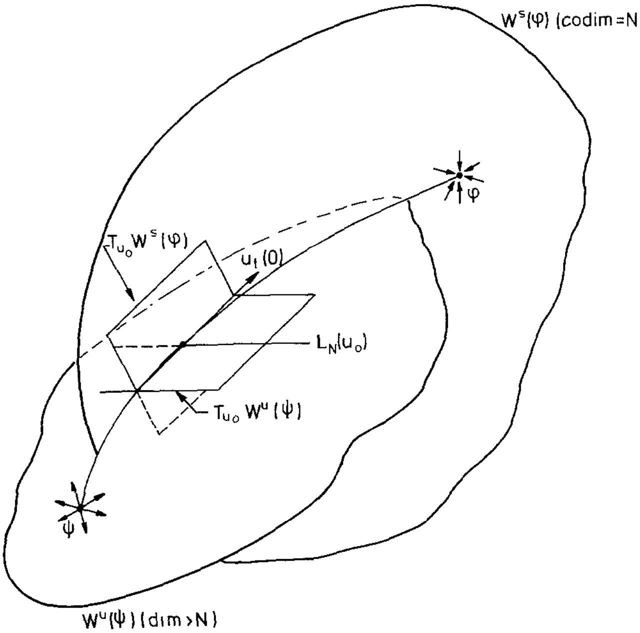

FIGURE 1

and thus

$$T_{u_0} W^u(\psi) + T_{u_0} W^s(\phi) = X$$

which completes the proof (see Fig. 1).

APPENDIX

In this section we deal with the asymptotic behaviour of solutions of Problem II. We shall do this in a slightly more general set up than in section two. Let E be a Hilbert space, and let A be an unbounded selfadjoint operator in E which satisfies the following hypotheses:

(A1) $\sigma(A)$ is bounded from below.

(A2) There exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$\beta_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$$

and $\sigma(A) \subset \bigcup_{k=0}^{\infty} (\beta_k, \alpha_{k+1})$ (i.e. $[\alpha_k, \beta_k] \cap \sigma(A)$ is empty for $k = 1, 2, \ldots$). We define $\gamma_k = \frac{1}{2}(\alpha_k + \beta_k)$ and $\delta_k = \frac{1}{2}(\beta_k - \alpha_k)$. The spectral projections of A will be denoted by $P(\lambda)$ ($\lambda \in \mathbb{R}$), and we shall write

$$P_k = P(\gamma_k) \qquad \text{and} \qquad Q_k = 1_E - P_k.$$

Furthermore, we assume that $\{B(t): t \geqslant 0\}$ is a family of bounded operators on E, which satisfy

(B1) $t \mapsto B(t)$ is Hölder continuous with respect to the operator norm on $\mathscr{L}(E)$.

(B2) $M =^{\text{def}} \sup_{t \geqslant 0} \|B(t)\|$ is finite, and in fact $\sup_{k \geqslant 1}(2M/\delta_k) < 1$. Condition (B2) says that the operators $B(t)$ are small compared to the gaps in the spectrum of A. Consider the linear problem

$$\begin{aligned} &u_t + Au = B(t)\,u \qquad 0 < t < \infty \\ &u(0) = u_0 \in E. \end{aligned} \tag{III}$$

It is well known that problem (III) determines a strongly continuous family $S(t)$, $t \geqslant 0$, of operators such that for any $u_0 \in E$ the unique solution

$$u(t) \in C^1((0, \infty); E) \cap C([0, \infty); E) \cap C((0, \infty); D(A))$$

is given by $u(t) = S(t)\,u_0$ (see [6, 10]). To study the large time behaviour of $u(t)$ we introduce the subspaces

$$V_k = \{u_0 \in E: \lim_{t \to \infty} e^{\gamma_k t} S(t)\,u_0 = 0\}, \qquad k = 1, 2, \ldots.$$

It is clear that $V_{k+1} \subset V_k$ for all $k \geqslant 1$.

Lemma 5. *If* (A1), (A2), (B1), *and* (B2) *hold, then*

(a) $Q(\gamma_k) = Q_k$: $V_k \to \text{Range}(Q_k)$ *is an isomorphism.*

(b) $\bigcap_{k \geqslant 1} V_k = \{0\}$, i.e., *for any nonzero* $u \in E$ *there is a* $k \geqslant 1$ *such that* $u \in V_k$ *but not* $u \in V_{k+1}$.

Remark. It follows from (b) that a solution of Problem (III) cannot decay to zero faster than exponentially.

Proof. Let $u_0 \in V_k$ be given, then $u(t) = S(t)\,u_0$ satisfies the integral equation

$$u(t) = e^{-tA} Qu(0) + \int_0^t e^{-(t-s)A} QB(s)\,u(s)\,ds$$

$$- \int_t^{\infty} e^{-(t-s)A} PB(s)\,u(s)\,ds,$$

where $Q = Q_k = Q(\gamma_k)$, and $P = P(\gamma_k)$ (so $P + Q = 1$). This implies that for any $k \geqslant 0$,

$$v(t) = e^{\gamma_k t} u(t)$$

satisfies the integral equation

$$v(t) = e^{t(\gamma - A)} Qu(0) + \int_0^\infty K(t-s)\, B(s)\, v(s)\, ds, \tag{IV}$$

where $\gamma = \gamma_k$ and

$$\begin{aligned} K(\tau) &= Qe^{\tau(\gamma - A)} && \text{when} \quad \tau > 0, \\ &= -Pe^{\tau(\gamma - A)} && \text{when} \quad \tau < 0. \end{aligned}$$

Conversely, if $v(t)$ is a solution of (IV) for which $\lim_{t \to \infty} v(t) = 0$ holds, then $u(t) = e^{-\gamma t} v(t)$ is a solution of (III).

Problem (IV) can be solved via the contraction mapping principle. To do this, consider the space

$$F = \{v \in C([0, \infty); E) \colon \lim_{t \to \infty} v(t) = 0\}$$

and define $L \colon F \to F$ by

$$Lv(t) = \int_0^\infty K(t-s)\, B(s)\, v(s)\, ds.$$

Note that F with the supremum norm is a Banach space. The linear operator L is a contraction on F, and hence bounded. Indeed, we have

$$\begin{aligned} \|Lv(t)\| &\leqslant \int_0^\infty \|K(t-s)\| \cdot \|B(s)\| \cdot \|v(s)\|\, ds \\ &\leqslant M \cdot \int_{-\infty}^\infty \|K(s)\|\, ds \cdot \sup_{s \geqslant 0} \|v(s)\| \end{aligned}$$

which shows that $\|L\| \leqslant M \cdot \int_{-\infty}^\infty \|K(s)\|\, ds$. Now remember that $K(s)$ is given by

$$\begin{aligned} K(s) &= \int_{\beta_k}^\infty e^{s(\gamma - \lambda)}\, dP(\lambda), && s > 0, \\ &= \int_{-\infty}^{\alpha_k} e^{s(\gamma - \lambda)}\, dP(\lambda), && s < 0, \end{aligned}$$

so $\|K(s)\| \leqslant e^{-\delta|s|}$, where $\delta = \delta_k$. Therefore

$$\|L\| \leqslant \frac{2M}{\delta_k} \leqslant \sup \frac{2M}{\delta_k} \cdot < 1. \tag{A.1}$$

Now let $q_0 = Qu(0) \in R(Q)$ be given, and define

$$q(t) = e^{(\gamma - A)t} q_0, \qquad t \geqslant 0. \tag{A.2}$$

Problem (IV) is then equivalent to

$$v(t) = q(t) + Lv(t) \qquad (t \geqslant 0), \tag{A.3}$$

which has the obvious solution

$$v(t) = \sum_{j=0}^{\infty} L^j q(t). \tag{A.4}$$

Let T_k: $R(Q) \to E$ be the mapping which assigns $v(0)$ to q_0, where $v(t)$ and q_0 are given by (A.2), (A.3), and (A.4). From (A.1) and (A.4) it follows that

$$\|T_k\| \leqslant \sum_{j=0}^{\infty} \left(\frac{2M}{\delta_k}\right)^j = \left(1 - \frac{2M}{\delta_k}\right)^{-1} < \infty. \tag{A.5}$$

One easily verifies that

$$Q_k \circ T_k = 1_{R(Q_k)}$$

and

$$T_k \circ Q_k = 1_{V_k}.$$

Hence Q_k: $V_k \to R(Q_k)$ is an isomorphism.

Next suppose that $u \in V_k$ for all $k \geqslant 1$. Then, using (A.5),

$$\begin{aligned} \|u\| &= \|T_k \circ Q_k u\| \\ &\leqslant \left(1 - \frac{2M}{\delta_k}\right)^{-1} \|Q_k u\|. \end{aligned}$$

However, $\lim_{k \to \infty} Q_k u = 0$, and $\sup_{k \geqslant 1} (1 - 2M/\delta_k)^{-1} < \infty$, so $u = 0$. This proves the second part of the lemma.

The result we need for the proof of Lemma 4 concerns the asymptotic direction of solutions of problem (III). Here we need an extra assumption:

(B3) $\lim_{t \to \infty} b(t) = 0$, where $b(t) = \|B(t)\|$.

Let $u(t) = S(t)\, u_0$ be a solution of (III), and define $x_k(t)$ and $y_k(t)$ by

$$x_k(t) = P_k u(t), \qquad y_k(t) = Q_k u(t).$$

We then have

LEMMA 6. *For any* $k \geqslant 1$,

$$\lim_{t \to \infty} \frac{\|x_k(t)\|}{\|y_k(t)\|} = 0 \qquad \textit{or } \infty,$$

where the quotient takes values in $[0, \infty]$.

Proof. Note that $x_k, y_k \in C^1((0, \infty); E) \cap C((0, \infty); D(A))$. Furthermore

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|x_k\|^2 &= (x_k, u'(t)) \\ &= -(x_k, Ax_k) + (x_k, B(t)\, u) \end{aligned}$$

and

$$\frac{1}{2} \frac{d}{dt} \|y_k\|^2 = -(y_k, Ay_k) + (y_k, B(t)\, u).$$

Hence, if $\lambda(t) = \log(\|y_k\| / \|x_k\|)$ then

$$2\lambda'(t) = \frac{(x_k, Ax_k)}{\|x_k\|^2} - \frac{(y_k, Ay_k)}{\|y_k\|^2} + \frac{(y_k, B(t)\, u)}{\|y_k\|^2} - \frac{(x_k, B(t) \cdot u)}{\|x_k\|^2}.$$

From

$$(x_k, Ax_k) \leqslant \alpha_k \|x_k\|^2$$

and

$$(y_k, Ay_k) \geqslant \beta_k \|y_k\|^2$$

it follows that

$$\begin{aligned} 2\lambda'(t) &\leqslant \alpha_k - \beta_k + b(t) \left\{ \frac{\|u\|}{\|y_k\|} + \frac{\|u\|}{\|x_k\|} \right\} \\ &\leqslant \alpha_k - \beta_k + b(t) \{2 + e^{-\lambda} + e^{\lambda}\}. \end{aligned}$$

Now suppose that λ does not converge to $+\infty$ or to $-\infty$ as t tends to infinity. Then there is a $\lambda_0 > 0$ such that for any $t_0 > 0$ there exists a $t_1 > t_0$ with

$$|\lambda(t_1)| \leqslant \lambda_0.$$

Choose t_0 so large that

$$b(t) \cdot \{2 + e^{2\lambda_0} + e^{-2\lambda_0}\} \leqslant \tfrac{1}{2}\delta_k$$

for all $t \geqslant t_0$. Then our estimate on $\lambda'(t)$ becomes

$$\lambda'(t) \leqslant -\tfrac{1}{2}\delta_k \qquad \text{when } t \geqslant t_0 \text{ and } |\lambda(t)| \leqslant 2\lambda_0.$$

So if $\lambda(t_1) \leqslant \lambda_0$ for some $t_1 \geqslant t_0$, then $\lambda(t)$ will decrease until it is less than $-\lambda_0$ and after that will never reach this value again, which is a contradiction. Therefore $\lim_{t\to\infty} \lambda(t) = \pm\infty$, i.e., $\lim_{t\to\infty} (\|x_k\|/\|y_k\|)$ is either zero or infinity.

In the proof of Lemma 4 we need the following consequence of Lemmas 5 and 6.

LEMMA 7. *Let $u_0 \neq 0$, and let $u(t)$ be a solution of Problem* (IV). *We set*

$$\hat{u}(t) = \frac{u(t)}{\|u(t)\|}.$$

Then there is a unique integer $k_0 \geqslant 1$, which depends on u_0 such that

(a) $\lim_{t\to\infty} \|P_k \hat{u}(t)\| = 0$ *for* $k < k_0$, 1 *for* $k \geqslant k_0$; *and*

(b) $u_0 \in V_{k_0 - 1} \setminus V_{k_0}$.

Proof. We apply Lemma 6 to the equality

$$\|P_k \hat{u}(t)\|^2 = \frac{\|x_k(t)\|^2}{\|x_k(t)\|^2 + \|y_k(t)\|^2}.$$

Hence $\lim_{t\to\infty} \|P_k \hat{u}(t)\|$ equals either one or zero. Since $\|P_k \hat{u}(t)\| \leqslant \|P_{k+1} \hat{u}(t)\|$ for all $k \geqslant 1$ there exist a unique k_0 (which may be infinite for the moment) such that (a) holds.

Next we show that

$$u_0 \in V_k \qquad \text{for} \quad k < k_0.$$

Observe that this implies, by Lemma 5, that k_0 is finite.

Let $k<k_0$ be given. Since

$$\frac{1}{2}\frac{d}{dt}(u, u) = -(u, Au)+(u, Bu)$$
$$= -(x_k, Ax_k)-(y_k, Ay_k)+(u, Bu)$$

we have

$$\frac{d}{dt}\log \|u\| = \{-(x_k, Ax_k)-(y_k, Ay_k)+(u, Bu)\}\cdot \|u\|^{-2}.$$

Hence

$$\frac{d}{dt}\log \|u\| \leqslant -\beta_0 \|P_k\hat{u}\|^2 - \beta_k \|\hat{u}-P_k\hat{u}\|^2 + b(t)$$
$$= -\beta_0 \|P_k\hat{u}\|^2 - \beta_k(1-\|P_k\hat{u}\|^2) + b(t),$$

and since $b(t)$ and $\|P_k\hat{u}(t)\|$ vanish as $t\to\infty$,

$$\limsup_{t\to\infty}\frac{d}{dt}\log \|u(t)\| \leqslant -\beta_k.$$

This implies that $\|u(t)\| \leqslant Ce^{-\gamma_k t}$ for some constant $C>0$ (recall that $\gamma_k<\beta_k$), and thus that $u_0\in V_k$, for all $k<k_0$.

It remains to prove that $u_0\notin V_{k_0}$. To do this we note that

$$\frac{d}{dt}\log \|x_k\| = \{-(x_k, Ax_k)+(x_k, B(t)u)\}\cdot \|x_k\|^{-2}$$
$$\geqslant -\alpha_k - b(t)\frac{\|u(t)\|}{\|x_k(t)\|}.$$

We put $k=k_0$ and let t tend to infinity. Since

$$\lim_{t\to\infty}\frac{\|x_{k_0}(t)\|}{\|u(t)\|}=1$$

we find that

$$\liminf_{t\to\infty}\frac{d}{dt}\log \|x_{k_0}(t)\| \geqslant -\alpha_{k_0} > -\gamma_{k_0}$$

which implies that $e^{+\gamma_{k_0}t}x_k(t)$ is unbounded. From the orthogonality of $x_k(t)$ and $y_k(t)$ we conclude that $e^{\gamma_{k_0}t}u(t)$ is also unbounded, and thus that $u_0 \notin V_{k_0}$. This completes the proof of Lemma 7.

Finally, let us indicate how this lemma can be applied to the situation in Lemma 4. Recall that the problem is

$$\begin{aligned} v_t &= v_{xx} + a(x, t)\, v, \qquad && 0<x<1, 0<t<\infty, \\ v(0, t) &= v(1, t) = 0, && 0<t<\infty, \end{aligned}$$

with $a(x, t) = f_u(x, u(x, t))$, and that $u(x, t)$ converges exponentially to $\phi(x)$, in $H_0^1(0, 1)$.

We choose $E = H_0^1(0, 1)$ and supply E with the following inner product:

$$(u, v)_E = \int_0^1 \{u'v' + (a(x)+k)\, uv\}\, dx$$

where $a(x) = f_u(x, \phi(x))$ and $k = \inf(a(x)\colon 0 \leqslant x \leqslant 1)$. This choice of the inner product makes the operator

$$-A = \left(\frac{d}{dx}\right)^2 + a(x)$$

$$D(A) = \{u \in H^3(0, 1)\colon u(0) = u''(0) = u(1) = u''(1) = 0\}$$

self-adjoint. It is natural now to define the operators $B(t)$ by

$$(B(t)\, u)(x) = (a(x, t) - a(x)) \cdot u(x).$$

The spectrum of the operator A is given by a sequence of simple eigenvalues $\lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and it is known that

$$\lambda_n = (n+1)^2\, \pi^2 + O(1) \qquad (n \to \infty)$$

(see [3]; here we use that A is a bounded perturbation of $(d/dx)^2$ with Dirichlet boundary conditions).

It is clear that conditions (A1), (A2), and (B1) are satisfied. Furthermore there exists a $t_0 > 0$ such that (B2) also holds, if we put

$$M = \sup(\|B(t)\|\colon t \geqslant t_0).$$

Now if $v(t)$ is a solution of problem II with nonzero initial value χ, then by backward uniqueness $v(t_0)$ will also be nonzero. Lemma 7 allows us to conclude that the asymptotic direction of $v(t)$ will be one of the eigenfunctions v_j.

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